

CALDERÓN-ZYGMUND OPERATORS IN THE BESSEL SETTING FOR ALL POSSIBLE TYPE INDICES

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ABSTRACT. In this paper we adapt the technique developed in [5] to show that many harmonic analysis operators in the Bessel setting, including maximal operators, Littlewood-Paley-Stein type square functions, multipliers of Laplace or Laplace-Stieltjes transform type and Riesz transforms are, or can be viewed as, Calderón-Zygmund operators for all possible values of type parameter λ in this context. This extends the results obtained recently in [3], which are valid only for a restricted range of λ .

1. INTRODUCTION AND PRELIMINARIES

Let $n \geq 1$ and $\lambda \in (-1/2, \infty)^n$. We consider the Bessel differential operator

$$\Delta_\lambda = -\Delta - \sum_{i=1}^n \frac{2\lambda_i}{x_i} \partial_{x_i},$$

where Δ stands for the Euclidean Laplacian in $\mathbb{R}_+^n = (0, \infty)^n$. The operator Δ_λ is symmetric and nonnegative in $C_c^\infty(\mathbb{R}_+^n) \subset L^2(\mathbb{R}_+^n, d\mu_\lambda)$, where $d\mu_\lambda$ is the doubling measure given by

$$d\mu_\lambda(x) = \prod_{i=1}^n x_i^{2\lambda_i} dx_i, \quad x \in \mathbb{R}_+^n.$$

It is well known that Δ_λ has a self-adjoint extension, here still denoted by Δ_λ , whose spectral decomposition is given via the Hankel transform, see [3] for details.

The semigroup $\{W_t^\lambda\}_{t>0}$ generated by $-\Delta_\lambda$ has the integral representation

$$W_t^\lambda f(x) = \int_{\mathbb{R}_+^n} W_t^\lambda(x, y) f(y) d\mu_\lambda(y), \quad x \in \mathbb{R}_+^n, \quad t > 0,$$

where the Bessel heat kernel is given by

(1.1)

$$W_t^\lambda(x, y) = \frac{1}{(2t)^n} \exp\left(-\frac{1}{4t}(|x|^2 + |y|^2)\right) \prod_{i=1}^n (x_i y_i)^{-\lambda_i+1/2} I_{\lambda_i-1/2}\left(\frac{x_i y_i}{2t}\right), \quad x, y \in \mathbb{R}_+^n, \quad t > 0,$$

with I_ν denoting the modified Bessel function of the first kind and order ν , cf. [9, p. 395]. Note that $\{W_t^\lambda\}$ is a symmetric diffusion semigroup in the sense of Stein's monograph ([6, p. 65]).

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In this setting the n -dimensional Hankel transform h_λ plays the same role as the Fourier transform in the Euclidean context. It is given by

$$h_\lambda f(x) = \int_{\mathbb{R}_+^n} \varphi_x^\lambda(y) f(y) d\mu_\lambda(y), \quad x \in \mathbb{R}_+^n,$$

with the kernel

$$\varphi_x^\lambda(y) = \prod_{i=1}^n (x_i y_i)^{-\lambda_i+1/2} J_{\lambda_i-1/2}(x_i y_i), \quad x, y \in \mathbb{R}_+^n,$$

where J_ν stands for the Bessel function of the first kind and order $\nu > -1$.

We investigate the following multi-dimensional Bessel operators defined initially either in $L^2(d\mu_\lambda)$ in the cases of (1)-(4), or in C^λ (the space of smooth L^2 -functions whose Hankel transform h_λ is also smooth and compactly supported) in the case of Riesz transforms (see [3, Section 4.4]).

- (1) The maximal operator

$$W_*^\lambda f = \|W_t^\lambda f\|_{L^\infty(dt)}.$$

- (2) Littlewood-Paley-Stein type mixed square functions

$$g_{m,k}^\lambda(f) = \|\partial^m \partial_t^k W_t^\lambda f\|_{L^2(t^{|m|+2k-1} dt)},$$

where $m \in \mathbb{N}^n$, $k \in \mathbb{N}$, $|m| + k > 0$.

- (3) Multipliers of Laplace transform type

$$T_{\mathcal{M}}^\lambda f = h_\lambda(\mathcal{M} h_\lambda f),$$

where $\mathcal{M}(z) = |z|^2 \int_0^\infty e^{-t|z|^2} \psi(t) dt$ with $\psi \in L^\infty(dt)$.

- (4) Multipliers of Laplace-Stieltjes transform type

$$T_{\mathcal{M}}^\lambda f = h_\lambda(\mathcal{M} h_\lambda f),$$

where $\mathcal{M}(z) = \int_{(0,\infty)} e^{-t|z|^2} d\nu(t)$ with ν being a complex Borel measure on $(0, \infty)$.

- (5) Riesz transforms of order m

$$R_m^\lambda f = \partial^m h_\lambda(|z|^{-|m|} h_\lambda f),$$

where $m \in \mathbb{N}^n$ and $|m| > 0$.

In [3] Betancor, Castro and Nowak showed that the above (vector-valued) operators, excluding (4), are Calderón-Zygmund in the sense of the space of homogeneous type $(\mathbb{R}_+^n, d\mu_\lambda, |\cdot|)$, but under the restriction $\lambda \in [0, \infty)^n$. The objective of this paper is to extend that result to the full range of $\lambda \in (-1/2, \infty)^n$, see Theorem 2.1 below. Typically, the main technical difficulty connected with the Calderón-Zygmund approach is to show the relevant kernel estimates. Here we follow the ideas of Nowak and Szarek [5], where they studied several Calderón-Zygmund operators in the Laguerre setting for all admissible multi-indices of type in that context.

Our results fit into the line of investigations concerning fundamental harmonic analysis operators associated with various discrete and continuous orthogonal expansions. Many of such operators were also widely examined in the one-dimensional Bessel context, see [2, Section 1] and [3, Section 1] for references. However, only recently the multi-dimensional Bessel situation was considered in [1, 2, 3].

The paper is organized as follows. Section 2 contains the statement of the main result (Theorem 2.1) and the reduction of its proof to showing the standard kernel estimates related to the Calderón-Zygmund theory. Furthermore, this section is concluded by a remark concerning operators analogous to (1)-(5) and associated with the square root of Δ_λ . Finally, in Section 3

various preparatory facts are gathered and then the proofs of the relevant kernel estimates are given.

Throughout the paper we use a standard notation with essentially all symbols pertaining to the space of homogeneous type $(\mathbb{R}_+^n, d\mu_\lambda, |\cdot|)$. However, for any unexplained symbol or notation we refer the reader to [3]. While writing estimates, we will use the notation $X \lesssim Y$ to indicate that $X \leq CY$ with a positive constant C independent of significant quantities. We will write $X \simeq Y$ when both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. MAIN RESULT

Let \mathbb{B} be a Banach space and let $K(x, y)$ be a kernel defined on $\mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \{(x, y) : x = y\}$ and taking values in \mathbb{B} . We say that $K(x, y)$ is a standard kernel in the sense of the space of homogeneous type $(\mathbb{R}_+^n, d\mu_\lambda, |\cdot|)$ if it satisfies the growth estimate

$$(2.1) \quad \|K(x, y)\|_{\mathbb{B}} \lesssim \frac{1}{\mu_\lambda(B(x, |x - y|))}$$

and the smoothness estimates

$$(2.2) \quad \|K(x, y) - K(x', y)\|_{\mathbb{B}} \lesssim \frac{|x - x'|}{|x - y|} \frac{1}{\mu_\lambda(B(x, |x - y|))}, \quad |x - y| > 2|x - x'|,$$

$$(2.3) \quad \|K(x, y) - K(x, y')\|_{\mathbb{B}} \lesssim \frac{|y - y'|}{|x - y|} \frac{1}{\mu_\lambda(B(x, |x - y|))}, \quad |x - y| > 2|y - y'|.$$

When $K(x, y)$ is scalar-valued, i.e. $\mathbb{B} = \mathbb{C}$, the difference bounds (2.2) and (2.3) are implied by the more convenient gradient estimate

$$(2.4) \quad |\nabla_{x,y} K(x, y)| \lesssim \frac{1}{|x - y| \mu_\lambda(B(x, |x - y|))}.$$

Notice that in these formulas, the ball $B(x, |y - x|)$ can be replaced by $B(y, |x - y|)$, in view of the doubling property of μ_λ .

A linear operator T assigning to each $f \in L^2(d\mu_\lambda)$ a measurable \mathbb{B} -valued function Tf on \mathbb{R}_+^n is a (vector-valued) Calderón-Zygmund operator in the sense of the space $(\mathbb{R}_+^n, d\mu_\lambda, |\cdot|)$ if

- (i) T is bounded from $L^2(d\mu_\lambda)$ to $L_{\mathbb{B}}^2(d\mu_\lambda)$,
- (ii) there exists a standard \mathbb{B} -valued kernel $K(x, y)$ such that

$$Tf(x) = \int_{\mathbb{R}_+^n} K(x, y) f(y) d\mu_\lambda(y), \quad \text{a.e. } x \notin \text{supp } f,$$

for every $f \in L_c^\infty(d\mu_\lambda)$, where $L_c^\infty(d\mu_\lambda)$ is the subspace of $L^\infty(d\mu_\lambda)$ of bounded measurable functions with compact supports.

The main result of the paper reads as follows.

Theorem 2.1. *Let $\lambda \in (-1/2, \infty)^n$, $m \in \mathbb{N}^n$, $k \in \mathbb{N}$ be such that $k + |m| > 0$, and assume that \mathcal{M} is as in (3) or as in (4) above. Then each of the operators*

$$W_*^\lambda, \quad g_{m,k}^\lambda, \quad T_{\mathcal{M}}^\lambda, \quad R_m^\lambda,$$

is a (vector-valued) Calderón-Zygmund operator in the sense of the space of homogeneous type $(\mathbb{R}_+^n, d\mu_\lambda, |\cdot|)$ associated with a Banach space \mathbb{B} , where \mathbb{B} is C_0 , $L^2(t^{|m|+2k-1}dt)$, \mathbb{C} , \mathbb{C} , respectively. Here C_0 denotes a separable subspace of $L^\infty(dt)$ consisting of all continuous functions on \mathbb{R}_+ which have finite limits as $t \rightarrow 0^+$ and vanish as $t \rightarrow \infty$.

Using the standard asymptotics for the Bessel function J_ν , $\nu > -1$, (cf. [9, Chapter III, Section 3.1 (8), Chapter VII, Section 7.21]),

$$J_\nu(z) \simeq z^\nu, \quad z \rightarrow 0^+, \quad J_\nu(z) = \mathcal{O}\left(\frac{1}{\sqrt{z}}\right), \quad z \rightarrow \infty,$$

we can estimate the one-dimensional kernel of the Hankel transform

$$|\varphi_{x_i}^{\lambda_i}(y_i)| \lesssim \begin{cases} 1, & x_i y_i \leq 1 \\ (x_i y_i)^{-\lambda_i}, & x_i y_i \geq 1. \end{cases}, \quad i = 1, \dots, n.$$

Combining this with the Bessel heat kernel representation (3.3) and Lemma 3.2 below, we see that [3, (13)] holds in fact for unrestricted $\lambda \in (-1/2, \infty)^n$. Then the same arguments as those given in [3] show that Lemmas 3.5, 3.7 and Remark 3.6 in [3] are actually valid for $\lambda \in (-1/2, \infty)^n$. Consequently, the methods developed in [3] to establish the L^2 -boundedness properties and kernels' associations for W_*^λ , $g_{m,k}^\lambda$, $T_{\mathcal{M}}^\lambda$ and R_m^λ , work also in this general case, provided that the standard estimates are true. Note that multipliers of Laplace-Stieltjes transform type were not treated in [3]. However, properties (i) and (ii) above for these operators can be shown essentially in the same way as for the Laplace transform type multipliers.

For the sake of clarity and completeness we recall from [3] the corresponding Calderón-Zygmund kernels and the related Banach spaces.

- (1) The kernel associated with the maximal operator is

$$\mathcal{W}^\lambda(x, y) = \{W_t^\lambda(x, y)\}_{t>0}, \quad \mathbb{B} = C_0 \subset L^\infty(dt).$$

- (2) The kernels associated with mixed square functions are

$$\mathcal{G}_{m,k}^\lambda(x, y) = \{\partial_t^k \partial_x^m W_t^\lambda(x, y)\}_{t>0}, \quad \mathbb{B} = L^2(t^{|m|+2k-1} dt),$$

where $m \in \mathbb{N}^n$ and $k \in \mathbb{N}$ are such that $|m| + k > 0$.

- (3) The kernels associated with Laplace transform type multipliers are

$$K_\psi^\lambda(x, y) = - \int_0^\infty \psi(t) \partial_t W_t^\lambda(x, y) dt, \quad \mathbb{B} = \mathbb{C},$$

where $\psi \in L^\infty(dt)$.

- (4) The kernels associated with Laplace-Stieltjes transform type multipliers are

$$K_\nu^\lambda(x, y) = \int_{(0,\infty)} W_t^\lambda(x, y) d\nu(t), \quad \mathbb{B} = \mathbb{C},$$

where ν are complex Borel measures on $(0, \infty)$.

- (5) The kernels associated with Riesz transforms are

$$R_m^\lambda(x, y) = \frac{1}{\Gamma(|m|/2)} \int_0^\infty \partial_x^m W_t^\lambda(x, y) t^{|m|/2-1} dt, \quad \mathbb{B} = \mathbb{C},$$

where $m \in \mathbb{N}^n$ is such that $|m| > 0$.

Thus to prove Theorem 2.1, it suffices to show the following.

Theorem 2.2. *Let $\lambda \in (-1/2, \infty)^n$. Then the kernels (1)–(5) listed above satisfy the standard estimates (2.1), (2.2) and (2.3) with the relevant Banach spaces \mathbb{B} .*

The proof of this theorem is the most technical part of this paper and is located in Section 3.

Remark 2.3. Let $\{P_t^\lambda\}_{t>0}$ be the Poisson-Bessel semigroup, which is generated by $-\sqrt{\Delta_\lambda}$. By the subordination principle,

$$P_t^\lambda f(x) = \int_0^\infty W_{t^2/(4u)}^\lambda f(x) \frac{e^{-u} du}{\sqrt{\pi u}}, \quad x \in \mathbb{R}_+^n, \quad t > 0.$$

We consider the maximal operator, Littlewood-Paley-Stein type square functions and multipliers of Laplace or Laplace-Stieltjes transform type based on this semigroup, see [3] for exact definitions. Then an analogous result to Theorem 2.1 is in force also for these operators. Basically, proving that for every $\lambda \in (-1/2, \infty)^n$ all these Poisson-type operators are (vector-valued) Calderón-Zygmund operators relies on the same arguments as those exposed in the incoming section, see [3], [5, Section 3] and [7, Section 4.3], thus we omit the details.

3. KERNEL ESTIMATES

This section delivers proofs of the standard estimates (2.1)-(2.3) for all the kernels under consideration. We extend the technique applied by Betancor, Castro and Nowak [3], which is valid for the restricted range of $\lambda \in [0, \infty)^n$. This method is based on Schlöfli's integral representation for the modified Bessel function I_ν , see [9, Chapter VI, Section 6.15] and [3, (7)],

$$(3.1) \quad I_\nu(z) = z^\nu \int_{[-1,1]} \exp(-zs) d\Omega_{\nu+1/2}(s), \quad z > 0, \quad \nu \geq -1/2,$$

where the measure Ω_η is a product of one-dimensional measures, $\Omega_\eta = \bigotimes_{i=1}^n \Omega_{\eta_i}$, with

$$d\Omega_{\eta_i}(s_i) = \frac{(1-s_i^2)^{\eta_i-1} ds_i}{\sqrt{\pi} 2^{\eta_i-1/2} \Gamma(\eta_i)}, \quad \eta_i > 0,$$

and in the limit case Ω_0 becomes the sum of unit point masses in 1 and -1 divided by $\sqrt{2\pi}$. Thus under the restriction $\lambda \in [0, \infty)^n$, the Bessel heat kernel can be written as, see [3, (8)],

$$(3.2) \quad W_t^\lambda(x, y) = \frac{1}{(2t)^{n/2+|\lambda|}} \int_{[-1,1]^n} \exp\left(-\frac{1}{4t}q(x, y, s)\right) d\Omega_\lambda(s), \quad x, y \in \mathbb{R}_+^n, \quad t > 0,$$

where $|\lambda| = \lambda_1 + \dots + \lambda_n$, and the function q is given by

$$q(x, y, s) = |x|^2 + |y|^2 + 2 \sum_{i=1}^n x_i y_i s_i, \quad x, y \in \mathbb{R}_+^n, \quad s \in [-1, 1]^n.$$

Following the ideas from [5], to express the Bessel heat kernel for the full range of $\lambda \in (-1/2, \infty)^n$, we use the recurrence relation for I_ν , see [9, Chapter III, Section 3.71],

$$I_\nu(z) = \frac{2(\nu+1)}{z} I_{\nu+1}(z) + I_{\nu+2}(z).$$

Combining this with (1.1) and (3.1) we arrive at the formula

$$(3.3) \quad W_t^\lambda(x, y) = \sum_{\varepsilon \in \{0,1\}^n} C_{\lambda,\varepsilon} t^{-n/2-|\lambda|-2|\varepsilon|} (xy)^{2\varepsilon} \int_{[-1,1]^n} \exp\left(-\frac{1}{4t}q(x, y, s)\right) d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s),$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}_+^n$, $(xy)^{2\varepsilon} = (x_1 y_1)^{2\varepsilon_1} \dots (x_n y_n)^{2\varepsilon_n}$, and $C_{\lambda,\varepsilon} = (2\lambda + \mathbf{1})^{\mathbf{1}-\varepsilon} 2^{-n/2-|\lambda|-2|\varepsilon|}$. This representation turns out to be convenient for our considerations connected with the Calderón-Zygmund theory.

To state the lemma below, and also for further use, it is convenient to introduce the following notation. Given $x, y \in \mathbb{R}_+^n$ and $\alpha \in \mathbb{R}^n$, we let

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$\begin{aligned}
xy &= (x_1 y_1, \dots, x_n y_n) \\
x \vee y &= (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\}) \\
x \leq y &\equiv x_i \leq y_i, \quad i = 1, \dots, n.
\end{aligned}$$

We will often neglect the set of integration $[-1, 1]^n$ in integrals against $d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s)$ and write shortly \mathfrak{q} instead of $q(x, y, s)$, provided that it does not lead to a confusion.

Lemma 3.1 ([5, Lemma 2.1]). *Let $\lambda \in (-1/2, \infty)^n$. Assume that $\xi, \kappa \in [0, \infty)^n$ are fixed and such that $\lambda + \xi + \kappa \in [0, \infty)^n$. Then*

$$(x + y)^{2\xi} \int \mathfrak{q}^{-n/2-|\lambda|-|\xi|} d\Omega_{\lambda+\xi+\kappa}(s) \lesssim \frac{1}{\mu_\lambda(B(x, |x-y|))},$$

uniformly in $x, y \in \mathbb{R}_+^n$, $x \neq y$.

This technical result, which is a natural generalization of [4, Proposition 5.9], is one of the main points in the whole method of proving kernel estimates. It establishes a relation between expressions involving certain integrals with respect to $d\Omega_{\lambda+\mathbf{1}+\varepsilon}(s)$, see (3.3), and the standard estimates for the space $(\mathbb{R}_+^n, d\mu_\lambda, |\cdot|)$.

It should be noted that for every $\lambda \in (-1/2, \infty)^n$ the μ_λ -measure of the ball $B(x, R)$ can be described by the same formula as in [3, Section 3], see [5, Lemma 2.2],

$$\mu_\lambda(B(x, R)) \simeq R^n \prod_{j=1}^n (x_j + R)^{2\lambda_j}, \quad x \in \mathbb{R}_+^n, \quad R > 0.$$

To estimate kernels defined via $W_t^\lambda(x, y)$ we will frequently use the following generalization of [3, Lemma 3.3].

Lemma 3.2. *Let $W \in \mathbb{R}$, $m, r \in \mathbb{N}^n$, $k \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}^n$. Then*

$$\begin{aligned}
&\left| \partial_t^k \partial_x^m \partial_y^r \left[t^W (xy)^{2\varepsilon} \exp\left(-\frac{1}{4t} \mathfrak{q}\right) \right] \right| \\
&\lesssim \sum_{\beta, \gamma \in \{0, 1, 2\}^n} x^{2\varepsilon-\beta\varepsilon} y^{2\varepsilon-\gamma\varepsilon} t^{W-k-(|m|+|\beta\varepsilon|+|r|+|\gamma\varepsilon|)/2} \exp\left(-\frac{1}{8t} \mathfrak{q}\right),
\end{aligned}$$

uniformly in $x, y \in \mathbb{R}_+^n$, $t > 0$ and $s \in [-1, 1]^n$.

Proof. First of all, we observe that

$$\partial_t^k \partial_x^m \partial_y^r \left[t^W (xy)^{2\varepsilon} \exp\left(-\frac{1}{4t} \mathfrak{q}\right) \right] = \partial_t^k \left[t^W \prod_{i=1}^n \partial_{x_i}^{m_i} \partial_{y_i}^{r_i} \left((x_i y_i)^{2\varepsilon_i} \exp\left(-\frac{1}{4t} \mathfrak{q}_i\right) \right) \right],$$

where $\mathfrak{q}_i = x_i^2 + y_i^2 + 2x_i y_i s_i$, $i = 1, \dots, n$. Given $i \in \{1, \dots, n\}$, we distinguish two cases. If $\varepsilon_i = 0$, then by [3, (10)] we know that

$$\begin{aligned}
&\partial_{x_i}^{m_i} \partial_{y_i}^{r_i} \exp\left(-\frac{1}{4t} \mathfrak{q}_i\right) \\
&= \sum_{\substack{0 \leq M_i \leq m_i \\ 0 \leq R_i \leq r_i}} P_{m_i, r_i, M_i, R_i}(s_i) t^{-(m_i+r_i+M_i+R_i)/2} (\partial_{x_i} \mathfrak{q}_i)^{M_i} (\partial_{y_i} \mathfrak{q}_i)^{R_i} \exp\left(-\frac{1}{4t} \mathfrak{q}_i\right),
\end{aligned}$$

where P_{m_i, r_i, M_i, R_i} are polynomials. On the other hand, when $\varepsilon_i = 1$, an application of Leibniz' rule and again [3, (10)] leads to

$$\partial_{x_i}^{m_i} \partial_{y_i}^{r_i} \left((x_i y_i)^2 \exp\left(-\frac{1}{4t} \mathfrak{q}_i\right) \right)$$

$$\begin{aligned}
&= \sum_{\beta_i, \gamma_i \in \{0,1,2\}} C_{m_i, r_i, \beta_i, \gamma_i} \chi_{\{\beta_i \leq m_i\}} \chi_{\{\gamma_i \leq r_i\}} x_i^{2-\beta_i} y_i^{2-\gamma_i} \partial_{x_i}^{m_i-\beta_i} \partial_{y_i}^{r_i-\gamma_i} \exp\left(-\frac{1}{4t} \mathbf{q}_i\right) \\
&= \sum_{\beta_i, \gamma_i \in \{0,1,2\}} x_i^{2-\beta_i} y_i^{2-\gamma_i} \sum_{\substack{0 \leq M_i \leq m_i-\beta_i \\ 0 \leq R_i \leq r_i-\gamma_i}} P_{m_i, r_i, \beta_i, \gamma_i, M_i, R_i}(s_i) t^{-(m_i-\beta_i+r_i-\gamma_i+M_i+R_i)/2} \\
&\quad \times (\partial_{x_i} \mathbf{q}_i)^{M_i} (\partial_{y_i} \mathbf{q}_i)^{R_i} \exp\left(-\frac{1}{4t} \mathbf{q}_i\right),
\end{aligned}$$

with $C_{m_i, r_i, \beta_i, \gamma_i} \in \mathbb{R}$ and $P_{m_i, r_i, \beta_i, \gamma_i, M_i, R_i}$ being polynomials. Hence,

$$\begin{aligned}
\partial_x^m \partial_y^r \left[(xy)^{2\varepsilon} \exp\left(-\frac{1}{4t} \mathbf{q}\right) \right] &= \sum_{\beta, \gamma \in \{0,1,2\}^n} x^{2\varepsilon-\beta\varepsilon} y^{2\varepsilon-\gamma\varepsilon} \\
&\times \sum_{\substack{0 \leq M \leq m-\beta\varepsilon \\ 0 \leq R \leq r-\gamma\varepsilon}} P_{m, r, \beta, \gamma, M, R, \varepsilon}(s) t^{-(|m|+|\beta\varepsilon|+|r|+|\gamma\varepsilon|+|M|+|R|)/2} (\partial_x \mathbf{q})^M (\partial_y \mathbf{q})^R \exp\left(-\frac{1}{4t} \mathbf{q}\right).
\end{aligned}$$

Now it remains to take derivatives with respect to t . By [3, (9)], it follows that

$$\begin{aligned}
\partial_t^k \left[t^W \partial_x^m \partial_y^r \left((xy)^{2\varepsilon} \exp\left(-\frac{1}{4t} \mathbf{q}\right) \right) \right] &= \sum_{\beta, \gamma \in \{0,1,2\}^n} x^{2\varepsilon-\beta\varepsilon} y^{2\varepsilon-\gamma\varepsilon} \sum_{\substack{0 \leq M \leq m-\beta\varepsilon \\ 0 \leq R \leq r-\gamma\varepsilon}} P_{m, r, \beta, \gamma, M, R, \varepsilon}(s) \\
&\times (\partial_x \mathbf{q})^M (\partial_y \mathbf{q})^R \sum_{0 \leq j \leq k} \alpha_{j, k, W, m, r, \beta, \gamma, M, R, \varepsilon} t^{W-k-j-(|m|+|\beta\varepsilon|+|r|+|\gamma\varepsilon|+|M|+|R|)/2} \mathbf{q}^j \exp\left(-\frac{1}{4t} \mathbf{q}\right),
\end{aligned}$$

for some $\alpha_{j, k, W, m, r, \beta, \gamma, M, R, \varepsilon} \in \mathbb{R}$.

Finally, using the estimates

$$|\partial_{x_i} \mathbf{q}| \lesssim \mathbf{q}^{1/2}, \quad |\partial_{y_i} \mathbf{q}| \lesssim \mathbf{q}^{1/2}, \quad i = 1, \dots, n,$$

and the fact that $\sup_{z \geq 0} z^\alpha e^{-z} < \infty$ for a fixed $\alpha \geq 0$, we get the asserted bound. \square

In the next lemma only values of $p \in \{1, 2, \infty\}$ will be needed for our purposes. However, using the result with $p \in (2, \infty)$ allows us to obtain the standard estimates for more general Littlewood-Paley-Stein type g -functions, for instance $g_{m, k, r}^{\lambda, W}$ investigated in [3].

Lemma 3.3. *Assume that $\lambda \in (-1/2, \infty)^n$, $1 \leq p \leq \infty$, $W \in \mathbb{R}$ and $C > 0$. Further, let $\varepsilon \in \{0, 1\}^n$ and $\vartheta, \varrho \in \{0, 1, 2\}^n$ be such that $\vartheta \leq 2\varepsilon$ and $\varrho \leq 2\varepsilon$. Given $u \geq 0$, we consider the function $\Upsilon_u: \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by*

$$\Upsilon_u(x, y, t) = t^{-n/2-|\lambda|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-W/p-u/2} x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \int \exp\left(-\frac{C\mathbf{q}}{t}\right) d\Omega_{\lambda+1+\varepsilon}(s),$$

where $W/p = 0$ for $p = \infty$. Then Υ_u satisfies the integral estimate

$$\|\Upsilon_u(x, y, t)\|_{L^p(t^{W-1}dt)} \lesssim \frac{1}{|x-y|^u} \frac{1}{\mu_\lambda(B(x, |y-x|))}$$

uniformly in $x, y \in \mathbb{R}_+^n$, $x \neq y$.

Proof. We assume that $p < \infty$. The case $p = \infty$ is similar and is left to the reader.

Applying Minkowski's integral inequality and then changing the variable $\mathbf{q}/t \mapsto \tau$ and using the inequality $|x-y|^2 \leq \mathbf{q}$, we obtain

$$\|\Upsilon_u(x, y, t)\|_{L^p(t^{W-1}dt)}$$

$$\begin{aligned}
&\leq x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \int \left\| t^{-n/2-|\lambda|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-W/p-u/2} \exp\left(-\frac{C\mathbf{q}}{t}\right) \right\|_{L^p(t^{W-1}dt)} d\Omega_{\lambda+1+\varepsilon}(s) \\
&= x^{2\varepsilon-\vartheta} y^{2\varepsilon-\varrho} \int \mathbf{q}^{-n/2-|\lambda|-2|\varepsilon|+|\vartheta|/2+|\varrho|/2-u/2} d\Omega_{\lambda+1+\varepsilon}(s) \\
&\quad \times \left(\int_0^\infty \tau^{p(n/2+|\lambda|+2|\varepsilon|-|\vartheta|/2-|\varrho|/2+u/2)-1} \exp(-Cp\tau) d\tau \right)^{1/p} \\
&\lesssim \frac{1}{|x-y|^u} (x+y)^{2(2\varepsilon-\vartheta/2-\varrho/2)} \int \mathbf{q}^{-n/2-|\lambda|-|2\varepsilon-\vartheta/2-\varrho/2|} d\Omega_{\lambda+1+\varepsilon}(s).
\end{aligned}$$

Now the required bound follows by means of Lemma 3.1 specified to $\xi = 2\varepsilon - \vartheta/2 - \varrho/2$ and $\kappa = \mathbf{1} - \varepsilon + \vartheta/2 + \varrho/2$. \square

The two lemmas below will be useful in justifying the smoothness estimates (2.2) and (2.3) when the corresponding kernel is not scalar valued ($\mathbb{B} \neq \mathbb{C}$).

Lemma 3.4 ([7, Lemma 4.5], [8, Lemma 4.3]). *Let $x, y, z \in \mathbb{R}_+^n$ and $s \in [-1, 1]^n$. Then*

$$\frac{1}{4}q(x, y, s) \leq q(z, y, s) \leq 4q(x, y, s),$$

provided that $|x - y| > 2|x - z|$. Similarly, if $|x - y| > 2|y - z|$ then

$$\frac{1}{4}q(x, y, s) \leq q(x, z, s) \leq 4q(x, y, s).$$

Lemma 3.5 ([8, Lemma 4.5]). *Let $\lambda \in (-1/2, \infty)^n$. We have*

$$\frac{1}{|z - y|\mu_\lambda(B(z, |z - y|))} \simeq \frac{1}{|x - y|\mu_\lambda(B(x, |x - y|))}$$

on the set $\{(x, y, z) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n : |x - y| > 2|x - z|\}$.

To be precise, in [8, Lemma 4.5] only restricted range of $\lambda \in [0, \infty)^n$ was allowed. However, the same arguments as those in [8] show the result in the general case.

Now we are in a position to prove Theorem 2.2. In the proof we always tacitly assume that passing with the differentiation in x_i , y_i and t under the integrals against $d\Omega_{\lambda+1+\varepsilon}(s)$, dt or $d\nu(t)$ is legitimate. In fact, such manipulations can be easily justified by using the estimates obtained along the proof of Theorem 2.2 and the dominated convergence theorem.

Proof of Theorem 2.2; the case of $\mathcal{W}^\lambda(x, y)$. Taking into account (3.3), the growth bound (2.1) for $\mathcal{W}^\lambda(x, y)$ is a straightforward consequence of Lemma 3.3 (specified to $u = 0$, $p = \infty$, $W = 1$, $C = 1/4$, $\vartheta = \varrho = 0$).

Next we focus on the smoothness conditions. By symmetry reasons it suffices to verify (2.2). An application of the Mean Value Theorem gives

$$|W_t^\lambda(x, y) - W_t^\lambda(x', y)| \leq |x - x'| \left| \nabla_x W_t^\lambda(x, y) \right|_{x=\theta},$$

where $\theta = \theta(t, x, x', y)$ is a convex combination of x and x' . Thus it is enough to show that

$$\left\| \left| \nabla_x W_t^\lambda(x, y) \right|_{x=\theta} \right\|_{L^\infty(dt)} \lesssim \frac{1}{|x - y|\mu_\lambda(B(x, |x - y|))}, \quad |x - y| > 2|x - x'|.$$

Differentiating (3.3) and then using sequently Lemma 3.2 (with $W = -n/2 - |\lambda| - 2|\varepsilon|$, $k = |r| = 0$, $m = e_j$, $j = 1, \dots, n$; here e_j is the j th coordinate vector in \mathbb{R}^n), the inequalities

$$(3.4) \quad \theta \leq x \vee x', \quad |x - \theta| \leq |x - x'|, \quad |x - x \vee x'| \leq |x - x'|,$$

and then Lemma 3.4 twice (first with $z = \theta$ and then with $z = x \vee x'$) we obtain

$$\begin{aligned} \left| \nabla_x W_t^\lambda(x, y) \right|_{x=\theta} &\lesssim \sum_{\varepsilon \in \{0,1\}^n} \sum_{\beta, \gamma \in \{0,1,2\}^n} \theta^{2\varepsilon - \beta\varepsilon} y^{2\varepsilon - \gamma\varepsilon} t^{-n/2 - |\lambda| - 2|\varepsilon| + |\beta\varepsilon|/2 + |\gamma\varepsilon|/2 - 1/2} \\ &\quad \times \int \exp\left(-\frac{1}{8t}q(\theta, y, s)\right) d\Omega_{\lambda+1+\varepsilon}(s) \\ &\leq \sum_{\varepsilon \in \{0,1\}^n} \sum_{\beta, \gamma \in \{0,1,2\}^n} (x \vee x')^{2\varepsilon - \beta\varepsilon} y^{2\varepsilon - \gamma\varepsilon} t^{-n/2 - |\lambda| - 2|\varepsilon| + |\beta\varepsilon|/2 + |\gamma\varepsilon|/2 - 1/2} \\ &\quad \times \int \exp\left(-\frac{1}{128t}q(x \vee x', y, s)\right) d\Omega_{\lambda+1+\varepsilon}(s), \end{aligned}$$

provided that $|x - y| > 2|x - x'|$. This, along with Lemma 3.3 (taken with $u = 1$, $p = \infty$, $W = 1$, $C = 1/128$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$) and Lemma 3.5 (with $z = x \vee x'$), gives the desired estimate. \square

Proof of Theorem 2.2; the case of $\mathcal{G}_{m,k}^\lambda(x, y)$. Combining Lemma 3.2 (applied with $W = -n/2 - |\lambda| - 2|\varepsilon|$, $|r| = 0$) with Lemma 3.3 (taken with $u = 0$, $p = 2$, $W = |m| + 2k$, $C = 1/8$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$) leads directly to the growth bound (2.1).

Proving the smoothness estimates we focus only on (2.2). The other bound is justified by analogous arguments. In view of the Mean Value Theorem, it suffices to verify that

$$\left\| \left| \nabla_x \partial_x^m \partial_t^k W_t^\lambda(x, y) \right|_{x=\theta} \right\|_{L^2(t^{|m|+2k-1}dt)} \lesssim \frac{1}{|x - y| \mu_\lambda(B(x, |x - y|))}, \quad |x - y| > 2|x - x'|,$$

where $\theta = \theta(t, x, x', y)$ is a convex combination of x and x' . Using sequentially Lemma 3.2 (specified to $W = -n/2 - |\lambda| - 2|\varepsilon|$, $|r| = 0$), the inequalities (3.4) and Lemma 3.4 twice (with $z = \theta$ and then with $z = x \vee x'$) we infer that

$$\begin{aligned} \left| \nabla_x \partial_x^m \partial_t^k W_t^\lambda(x, y) \right|_{x=\theta} &\lesssim \sum_{\varepsilon \in \{0,1\}^n} \sum_{\beta, \gamma \in \{0,1,2\}^n} (x \vee x')^{2\varepsilon - \beta\varepsilon} y^{2\varepsilon - \gamma\varepsilon} t^{-n/2 - |\lambda| - 2|\varepsilon| - k - (|m| - |\beta\varepsilon| - |\gamma\varepsilon|)/2 - 1/2} \\ &\quad \times \int \exp\left(-\frac{1}{128t}q(x \vee x', y, s)\right) d\Omega_{\lambda+1+\varepsilon}(s). \end{aligned}$$

Hence, with the aid of Lemma 3.3 (applied with $u = 1$, $p = 2$, $W = |m| + 2k$, $C = 1/128$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$) and Lemma 3.5 (taken with $z = x \vee x'$), we arrive at the conclusion. \square

Proof of Theorem 2.2; the case of $K_\psi^\lambda(x, y)$. The growth condition is a simple consequence of Lemma 3.2 (specified to $W = -n/2 - |\lambda| - 2|\varepsilon|$, $k = 1$ and $|m| = |r| = 0$), the fact that ψ is bounded, and Lemma 3.3 (taken with $u = 0$, $p = 1$, $W = 1$, $C = 1/8$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$).

We pass to proving the gradient estimate (2.4). Since $\psi \in L^\infty(dt)$, it is enough to check that

$$\left\| \nabla_{x,y} \partial_t W_t^\lambda(x, y) \right\|_{L^1(dt)} \lesssim \frac{1}{|x - y| \mu_\lambda(B(x, |x - y|))}, \quad x \neq y.$$

This, however, follows by combining Lemma 3.2 (specified to $W = -n/2 - |\lambda| - 2|\varepsilon|$, $k = 1$ and $m = e_j$, $|r| = 0$ or $|m| = 0$, $r = e_j$, $j = 1, \dots, n$) with Lemma 3.3 (applied with $u = 1$, $p = 1$, $W = 1$, $C = 1/8$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$). \square

Proof of Theorem 2.2; the case of $K_\nu^\lambda(x, y)$. Since the measure ν is complex (in particular, its total variation is finite), in order to prove the standard estimates it suffices to verify that

$$\left\| W_t^\lambda(x, y) \right\|_{L^\infty(dt)} \lesssim \frac{1}{\mu_\lambda(B(x, |x - y|))}, \quad x \neq y,$$

$$\left\| |\nabla_{x,y} W_t^\lambda(x, y)| \right\|_{L^\infty(dt)} \lesssim \frac{1}{|x - y| \mu_\lambda(B(x, |x - y|))}, \quad x \neq y.$$

The first bound here is just the growth condition for $\mathcal{W}^\lambda(x, y)$, which is already justified. The second one is implicitly contained in the proof of the smoothness estimates for $\mathcal{W}^\lambda(x, y)$. \square

Proof of Theorem 2.2; the case of $R_m^\lambda(x, y)$. The growth condition is obtained by using Lemma 3.2 (specified to $W = -n/2 - |\lambda| - 2|\varepsilon|$, $k = |r| = 0$) and then Lemma 3.3 (with $u = 0$, $p = 1$, $W = |m|/2$, $C = 1/8$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$).

To prove the gradient bound (2.4), it suffices to show that

$$\left\| |\nabla_{x,y} \partial_x^m W_t^\lambda(x, y)| \right\|_{L^1(t^{|m|/2-1}dt)} \lesssim \frac{1}{|x - y| \mu_\lambda(B(x, |x - y|))}, \quad x \neq y.$$

This, however, follows by using Lemma 3.2 (taken with $W = -n/2 - |\lambda| - 2|\varepsilon|$, $k = |r| = 0$) and then Lemma 3.3 (applied with $u = 1$, $p = 1$, $W = |m|/2$, $C = 1/8$, $\vartheta = \beta\varepsilon$ and $\varrho = \gamma\varepsilon$). \square

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